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Inhomogeneous Contact Processes on Trees

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We consider an inhomogeneous contact process on a tree $\mathbb{T}_k$ of degree k, where the infection rate at any site is λ, the death rate at any site in $S \subset \mathbb{T}_k$ is δ (with $0 < \delta \leqslant 1$) and that at any site in $\mathbb{T}_k - S$ is 1. Denote by $\lambda_c(\mathbb{T}_k)$ the critical value for the *homogeneous* model (i.e., $\delta = 1$) on $\mathbb{T}_k$ and by $\theta(\delta, \lambda)$ the survival probability of the inhomogeneous model on $\mathbb{T}_k$. We prove that when $k > 4$, if $S = \mathbb{T}_\sigma$, a subtree embedded in $\mathbb{T}_k$, with $1 \leqslant \sigma \leqslant \sqrt{k}$, then there exists δ_c^σ strictly between $\lambda_c(\mathbb{T}_k)/\lambda_c(\mathbb{T}_\sigma)$ and 1 such that $\theta(\delta, \lambda_c(\mathbb{T}_k)) = 0$ when $\delta > \delta_c^\sigma$ and $\theta(\delta, \lambda_c(\mathbb{T}_k)) > 0$ when $\delta < \delta_c^\sigma$; if $S = \{o\}$, the origin of $\mathbb{T}_k$, then $\theta(\delta, \lambda_c(\mathbb{T}_k)) = 0$ for any $\delta \in (0, 1)$.

KEY WORDS: Contact process; inhomogeneity; trees.

1. INTRODUCTION

The contact process on a lattice $\mathbb{L}$ can be defined by the usual graphical representation as follows. Consider the space $\mathbb{L} \times [0, \infty)$, in which $\mathbb{L}$ represents the spatial component and $[0, \infty)$ represents time. Along each vertical time line $\{x\} \times [0, \infty)$, where x is a site in $\mathbb{L}$, is positioned a Poisson process of points, called deaths, with density $\delta(x)$; between each ordered pair $\{x_1\} \times [0, \infty)$ and $\{x_2\} \times [0, \infty)$ of nearest neighbor lines, there is a Poisson process, with density λ, of bonds oriented in the direction x_1 to x_2. All these Poisson processes are taken to be independent of each other. In this note we consider the contact process on $\mathbb{T}_k$, a tree with degree k (i.e., each site of $\mathbb{T}_k$ has exactly $k+1$ neighbors). For results on the *homogeneous* contact process on trees (i.e., $\delta(x) = 1$ for all $x \in \mathbb{T}_k$), see for example refs. 7, 11, and 12. We are interested in the following two cases of *inhomogeneous* models:

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Case (a). $\delta(x)=\delta$ for any $x\in\mathbb{T}_\sigma$ and $\delta(x)=1$ for any $x\in\mathbb{T}_k-\mathbb{T}_\sigma$, where $\mathbb{T}_\sigma$ is a subtree of degree σ (with $1\leqslant\sigma<k$) embedded in $\mathbb{T}_k$.

Case (b). $\delta(o)=\delta$ and $\delta(x)=1$ for any $x\in\mathbb{T}_k-\{o\}$, where o is the origin of $\mathbb{T}_k$.

We write $P_{\delta,\lambda}$ for the resulting probability measure and $E_{\delta,\lambda}$ for the corresponding expectation. When $\delta=1$, this corresponds to the homogeneous model, for which we will simply write P_λ rather than $P_{1,\lambda}$. A Point (x, t_1) in $\mathbb{L}\times[0,\infty)$ is said to be connected to another point (y, t_2) (with $t_1\leqslant t_2$) if there is a path from (x, t_1) to (y, t_2) using vertical line segments traversed in the upward direction and oriented horizontal bonds and traversing no points of death. We denote this event by $(x, t_1)\to(y, t_2)$.

Let ξ_t be the set of sites x in $\mathbb{L}$ such that the origin $(o, 0)$ of $\mathbb{L}\times[0,\infty)$ is connected to (x, t). If we treat the contact process in the usual way as a model for the spread of infection, then ξ_t is the set of infected sites at time t when initially only the origin is infected. Let

$$C(x,t)=\{(y,s):(x,t)\to(y,s)\}$$

denote the cluster of (x, t). We write C for $C(o, 0)$. Define the survival probability to be

$$\theta(\delta,\lambda)=P_{\delta,\lambda}(\xi_t\neq\varnothing\text{ for all }t\geqslant 0) \tag{1.1}$$

where $\varnothing$ denotes the empty set. Then the critical value (for the homogeneous model) is defined by

$$\lambda_c=\lambda_c(\mathbb{L})=\inf\{\lambda:\theta(1,\lambda)>0\}. \tag{1.2}$$

For the contact process on $\mathbb{Z}^d$, the analogues of cases (a) and (b) are:

Case (a′). $\delta(x)=\delta$ for any $x\in\{0\}^{d-s}\times\mathbb{Z}^s$ and $\delta(x)=1$ for any $x\in\mathbb{Z}^d-(\{0\}^{d-s}\times\mathbb{Z}^s)$, where $1\leqslant s\leqslant d-1$.

Case (b′). $\delta(o)=\delta$ and $\delta(x)=1$ for any $x\in\mathbb{Z}^d-\{o\}$, where o denotes the origin of $\mathbb{Z}^d$.

Madras, Schinazi and Schonmann[8] considered case (b′) and proved, among many other things, that if $\lambda<\lambda_c$, then

$$\theta(\delta,\lambda)=0 \qquad \text{for any} \quad \delta\in(0,1]. \tag{1.3}$$

They conjectured that

$$\theta(\delta,\lambda_c)=0 \qquad \text{for any} \quad \delta\in(0,1]. \tag{1.4}$$

Note that (1.4) (for $1 < \delta < 1$) is stronger than

$$\theta(1, \lambda_c) = 0, \tag{1.5}$$

which is a fundamental result of Bezuidenhout and Grimmett.[4] Newman and Wu[10] considered case (a′) and conjectured that (for $1 \leqslant s \leqslant d-1$) there exists a critical point δ_c^s in $(\lambda_c(\mathbb{Z}^d)/\lambda_c(\mathbb{Z}^s), 1)$ for δ such that

$$\begin{cases} \theta(\delta, \lambda_c) = 0 & \text{if } \delta > \delta_c^s \\ \theta(\delta, \lambda_c) > 0 & \text{if } \delta < \delta_c^s \end{cases} \tag{1.6}$$

They also found some sufficient conditions for conjectures (1.4) and (1.6) to be true in high dimensions. These sufficient conditions can be verified using the infrared bound for the contact process on $\mathbb{Z}^d$. However the infrared bound for the contact process on $\mathbb{Z}^d$ has not been rigorously proved, although it is expected to be true when $d > 4$. Conjectures (1.4) and (1.6) remain open for any dimension.

In this note, we consider the inhomogeneous contact process on trees, i.e., cases (a) and (b). We will prove the analogues of conjectures (1.4) and (1.6) for trees when the degree k is above four, using an estimate of the connectivity function of the contact process on trees proved in ref. 13 (see (2.15) below). We remark that the argument presented here does not work for the contact process on $\mathbb{Z}^d$ because we do not have an estimate like (2.15) on $\mathbb{Z}^d$. Recall that $\lambda_c(\mathbb{L})$ is the critical value of λ for the (homogeneous) contact process on $\mathbb{L}$. We write λ_c for $\lambda_c(\mathbb{T}_k)$ when it does not cause confusion.

Theorem 1. In case (a), if $k > 4$ and $1 \leqslant \sigma \leqslant \sqrt{k}$, then there exists δ_c^σ in $(\lambda_c/\lambda_c(\mathbb{T}_\sigma), 1)$ such that

$$\begin{cases} \theta(\delta, \lambda_c) = 0 & \text{if } \delta > \delta_c^\sigma \\ \theta(\delta, \lambda_c) > 0 & \text{if } \delta < \delta_c^\sigma \end{cases} \tag{1.7}$$

Theorem 2. In case (b), if $k > 4$, then

$$\theta(\delta, \lambda_c) = 0 \qquad \text{for any} \quad \delta \in (0, 1] \tag{1.8}$$

Note that it has been proved that $\theta(1, \lambda_c) = 0$ for any $k \geqslant 2$ by Morrow, Schinazi and Zhang[9] and for $k = 1$ by Bezuidenhout and Grimmett[4]—a tree with $k = 1$ is just the one dimensional lattice $\mathbb{Z}$.

We expect that Theorem 1 is true for any $k \geqslant 2$ and $1 \leqslant \sigma \leqslant k-1$, and Theorem 2 is true for any $k \geqslant 1$. At the present our argument does not work when $2 \leqslant k \leqslant 4$ or when $\sqrt{k} < \sigma \leqslant k-1$ for Theorem 1 and when $1 \leqslant k \leqslant 4$ for Theorem 2. On the other hand, it is not hard to see that *in* *case* (b), for any $k \geqslant 2$ there exists $\varepsilon > 0$ such that $\theta(\delta, \lambda_c) = 0$ for any $\delta \in (1-\varepsilon, 1]$, using the fact that for the homogeneous model, when the infection rate is a little above λ_c, the infection will eventually leave any finite region (see refs. 7, 11, 12). Finally we remark that the analogous results of Theorems 1 and 2 for independent percolation and oriented percolation on $\mathbb{T}_k \times \mathbb{Z}$ (see refs. 6 and 13) can be obtained by the same argument presented here.

2. PROOF OF THEOREMS

Proof of Theorem 1. First of all, it is not hard to see that if $\delta < \lambda_c/\lambda_c(\mathbb{T}_\sigma)$ then $\theta(\delta, \lambda_c) > 0$, since if $\delta < \lambda_c/\lambda_c(\mathbb{T}_\sigma)$ then $\lambda_c/\delta > \lambda_c(\mathbb{T}_\sigma)$ and hence the infection already survives on the subtree $\mathbb{T}_\sigma$ embedded in $\mathbb{T}_k$. Thus $\delta_c^\sigma \geqslant \lambda_c/\lambda_c(\mathbb{T}_\sigma)$. The strict inequality $\delta_c^\sigma > \lambda_c/\lambda_c(\mathbb{T}_\sigma)$ then follows from Theorem 1 of Aizenman and Grimmett[1] applied to the contact process (see pp. 826–827 of ref. 1), since the positive infection rate on the sites of $\mathbb{T}_k - \mathbb{T}_\sigma$ provides an essential enhancement for the contact process on $\mathbb{T}_\sigma$. On the other hand, to show $\delta_c^\sigma < 1$, one needs to prove that there exists $\varepsilon > 0$ such that

$$\theta(1-\varepsilon, \lambda_c) = 0 \tag{2.1}$$

Let

$$C^* = C(o, 0) \cap (\mathbb{T}_\sigma \times \mathbb{R}^+) = \{(x, t) : x \in \mathbb{T}_\sigma, (o, 0) \to (x, t) \text{ in } \mathbb{T}_k \times \mathbb{R}^+\} \tag{2.2}$$

be the intersection of the cluster of the origin with $\mathbb{T}_\sigma \times \mathbb{R}^+$ ($\mathbb{R}^+ = [0, \infty)$). For $A_x \subset \{x\} \times \mathbb{R}^+$, we denote by $\|A_x\|$ the Lebesgue measure of A_x and for $A \subset \mathbb{T}_k \times \mathbb{R}^+$, we define

$$\|A\| = \sum_{x \in \mathbb{T}_k} \|A \cap (\{x\} \times \mathbb{R}^+)\| \tag{2.3}$$

Then we define the expected infection time in $\mathbb{T}_\sigma$ by

$$\chi(\delta) \equiv E_{\delta, \lambda_c} \|C^*\| = \sum_{x \in \mathbb{T}_\sigma} \int_0^\infty P_{\delta, \lambda_c}((o, 0) \to (x, t))\, dt \tag{2.4}$$

The idea of proving (2.1) is to show that

$$\chi(1-\varepsilon) < \infty. \tag{2.5}$$

To see how (2.1) follows from (2.5), we first show that (2.5) implies that almost surely $C(o, 0)$ will never intersect $\mathbb{T}_\sigma \times \mathbb{R}^+$ when t is large (see ref. (2.10) below). This is not quite obvious since it could be the case that $C(o, 0)$ intersects $\mathbb{T}_\sigma \times \mathbb{R}^+$ infinitely often (at arbitrarily large t) but in such a way that it hits $\mathbb{T}_\sigma \times \mathbb{R}^+$ (for a very short period) and then leaves, and then comes back at some later time and hits and leaves again, and so on forever. It is not hard to show that

$$\sum_{x \in \mathbb{T}_\sigma} \sum_{n=0}^{\infty} P_{\delta, \lambda_c}((o, 0) \to \{x\} \times [n, n+1)) \leqslant c \sum_{x \in \mathbb{T}_\sigma} \int_0^\infty P_{\delta, \lambda_c}((o, 0) \to (x, t))\, dt \tag{2.6}$$

where $\{x\} \times [n, n+1)$ is an unit interval on the line $\{x\} \times \mathbb{R}^+$ and $c = 1/P_{\delta, \lambda_c}$ (no death in $\{x\} \times [n, n+2)$) ($\leqslant e^2$). To see this, observe that for any $t \in [n+1, n+2)$

$$\begin{aligned} P_{\delta, \lambda_c}((o, 0) \to \{x\} \times [n, n+1), &\text{ no death in } \{x\} \times [n, n+2)) \\ &\leqslant P_{\delta, \lambda_c}((o, 0) \to (x, t)) \end{aligned} \tag{2.7}$$

which implies by the FKG inequalities (see ref. 5) that

$$P_{\delta, \lambda_c}((o, 0) \to \{x\} \times [n, n+1)) \leqslant c P_{\delta, \lambda_c}((o, 0) \to (x, t)) \tag{2.8}$$

Inequality (2.6) then follows from integrating (2.8) (with respect to t) over $[n+1, n+2)$ and then sum over n. Define

$$Int(C^*) = \{\{x\} \times [n, n+1) : (\{x\} \times [n, n+1)) \cap C^* \neq \varnothing\} \tag{2.9}$$

to be the set of unit intervals which C^* intersects. Write $|Int(C^*)|$ for the number of elements in $Int(C^*)$. Then by (2.5) and (2.6)

$$E_{1-\varepsilon, \lambda_c} |Int(C^*)| \leqslant c\chi(1-\varepsilon) < \infty \tag{2.10}$$

Write

$$\begin{aligned} \{\xi_t \neq \varnothing, \forall t \geqslant 0\} = &\{\xi_t \neq \varnothing, \forall t \geqslant 0 \text{ and } |Int(C^*)| < \infty\} \\ &\cup \{xi_t \neq \varnothing, \forall t \geqslant 0 \text{ and } |Int(C^*)| = \infty\} \end{aligned} \tag{2.11}$$

Then from (2.10),

$$P_{1-\varepsilon, \lambda_c}(\xi_t \neq \varnothing, \forall t \geqslant 0 \text{ and } |Int(C^*)| = \infty) = 0 \tag{2.12}$$

Moreover

$$\begin{aligned}
&P_{1-\varepsilon,\lambda_c}(\xi_t \neq \varnothing, \forall t \geqslant 0 \text{ and } |Int(C^*)| < \infty)\\
&\quad \leqslant P_{1-\varepsilon,\lambda_c}(\exists \text{ a positive integer } n \text{ and a site } x \in \mathbb{T}_k - \mathbb{T}_\sigma\\
&\qquad \text{such that } (o, 0) \to (x, n),\\
&\qquad \text{and infection initiated at } (x, n) \text{ survives forever within } \mathbb{T}_k - \mathbb{T}_\sigma)\\
&\quad \leqslant \sum_{n=0}^{\infty} \sum_{x \in \mathbb{T}_k - \mathbb{T}_\sigma} P_{1-\varepsilon,\lambda_c}(\text{infection initiated at } (x, n)\\
&\qquad \text{survives forever within } \mathbb{T}_k - \mathbb{T}_\sigma)\\
&\quad = \sum_{n=0}^{\infty} \sum_{x \in \mathbb{T}_k - \mathbb{T}_\sigma} P_{1,\lambda_c}(\text{infection initiated at } (x, n)\\
&\qquad \text{survives forever within } \mathbb{T}_k - \mathbb{T}_\sigma)\\
&\quad \leqslant \sum_{n=0}^{\infty} \sum_{x \in \mathbb{T}_k - \mathbb{T}_\sigma} P_{1,\lambda_c}(\text{infection initiated at } (x, n)\\
&\qquad \text{survives forever within } \mathbb{T}_k)\\
&\quad = \sum_{n=0}^{\infty} \sum_{x \in \mathbb{T}_k - \mathbb{T}_\sigma} \theta(1, \lambda_c) = 0. \qquad (2.13)
\end{aligned}$$

Where at the last step we use the result that $\theta(1, \lambda_c) = 0$. Combining (2.11)–(2.13) implies (2.1).

We next prove (2.5). We first show that

$$\chi(1) \equiv \sum_{x \in \mathbb{T}_\sigma} \int_0^\infty P_{1,\lambda_c}((o, 0) \to (x, t))\, dt < \infty \tag{2.14}$$

when $k > 4$ and $1 \leqslant \sigma \leqslant \sqrt{k}$ (this is where we need the unpleasant assumption). From Lemma 2 in ref. 13 we have

$$\int_0^\infty P_{1,\lambda}((o, 0) \to (x, t))\, dt \leqslant \frac{1}{1 - 2\sqrt{k}\,\lambda} (2\lambda)^{|x|} \tag{2.15}$$

when $\lambda < 1/(2\sqrt{k})$, where $|x|$ is the graph distance from $x \in \mathbb{T}_k$ to the origin of $\mathbb{T}_k$. Since it is also known that $2\sqrt{k}\,\lambda_c < 1$ for $k > 4$ (see Theorem 2.2

in ref. 11 or the proof of Corollary 1 in ref. 13), we have from (2.14) and (2.15) that

$$\chi(1) \leqslant \frac{1}{1-2\sqrt{k}\,\lambda_c} \sum_{x \in \mathbb{T}_\sigma} (2\lambda_c)^{|x|}$$
$$= \frac{1}{1-2\sqrt{k}\,\lambda_c}\left[1+\sum_{n=1}^{\infty}(\sigma+1)\,\sigma^{n-1}(2\lambda_c)^n\right] \tag{2.16}$$

where the equality uses the isotropy of the tree $\mathbb{T}_\sigma$. The RHS of (2.16) is finite when $(1\leqslant)\ \sigma \leqslant \sqrt{k}$.

Finally we show that $\chi(1)<\infty$ implies $\chi(1-\varepsilon)<\infty$ for some (small) $\varepsilon>0$. It is proved in ref. 3 (see Proposition 3.2 there) that

$$(0\leqslant)\,\frac{d}{d\lambda}\,[E_{1,\lambda}\,\|C(o,0)\|] \leqslant (k+1)[E_{1,\lambda}\,\|C(o,0)\|]^2 \tag{2.17}$$

which extends the analogous differential inequality obtained by Aizenman and Newman[2] for percolation to the contact process. The same argument can be used to prove that

$$(0\leqslant)\,-\frac{d}{d\delta}\,\chi(\delta) \leqslant [\chi(\delta)]^2 \tag{2.18}$$

We note that there is no factor of $k+1$ in (2.18) because the derivative there is with respect to the death rate (in $\mathbb{T}_\sigma$) rather than the infection rate λ, and there is a negative sign because $\chi(\delta)$ is a decreasing function of δ. Write (2.18) as

$$\frac{d}{d\delta}\,[\chi(\delta)]^{-1} \leqslant 1 \tag{2.19}$$

and integrate it over $[1-\varepsilon, 1]$, we have that

$$\chi(1)^{-1}-\chi(1-\varepsilon)^{-1} \leqslant \varepsilon \qquad \text{or} \quad \chi(1-\varepsilon) \leqslant \frac{\chi(1)}{1-\varepsilon\chi(1)} \tag{2.20}$$

for $0<\varepsilon<\chi(1)^{-1}$. This completes the proof of the Theorem. ∎

Proof of Theorem 2. For technical reasons, we extend the time coordinate from $[0, \infty)$ to $\mathbb{R}=(-\infty, \infty)$. It should be clear that the graphical representation can be extended to $\mathbb{T}_k \times \mathbb{R}$. We write $\{o\} \times t$ instead of (o, t) for a point on $\{o\} \times \mathbb{R}$. We use $\{o\} \times (a, b] \to \{o\} \times (a', b']$ to denote the event that $\{o\}\times t \to \{o\}\times t'$ for some $a<t\leqslant b$ and $a'<t'\leqslant b'$. We call a point $\{o\}\times n$ of $\{o\}\times\mathbb{R}$ (with n an integer) a breakpoint if $\{o\}\times(-\infty, n-1] \nrightarrow \{o\}\times(n,\infty)$. Denote by "$\{o\}\times(\ell-1,\ell] \nrightarrow \{o\}\times(m,m+1]$ in $(\mathbb{T}_k-\{o\})\times\mathbb{R}$" the complement of the event that there exist $\{o\}\times t\in\{o\}\times(\ell-1,\ell]$ and $\{o\}\times u\in\{o\}\times(m,m+1]$ such that $\{o\}\times t$ is connected to $\{o\}\times u$ by a path γ which intersects $\{o\}\times\mathbb{R}$ only at $\{o\}\times t$ and $\{o\}\times u$. Note that this event is independent of the Poisson process of deaths on $\{o\}\times\mathbb{R}$. Let B be the event that there is a death in the interval $\{o\}\times(0,1)$ and there is no bond leaving or entering $\{o\}\times(0,1]$. Then by the FKG inequalities we have for any $0<\delta<1$ that

$$\begin{aligned}
&P_{\delta,\lambda_c}(\{o\}\times 1 \text{ is a breakpoint})\\
&\quad\geqslant P_{\delta,\lambda_c}(B \text{ occurs, and } \{o\}\times(-\infty,0] \nrightarrow \{o\}\times(1,\infty)\\
&\qquad \text{in } (\mathbb{T}_k-\{o\})\times\mathbb{R})\\
&\quad\geqslant P_{\delta,\lambda_c}(B)\prod_{\ell\leqslant 0,\, m\geqslant 1} P_{\delta,\lambda_c}(\{o\}\times(\ell-1,\ell] \nrightarrow \{o\}\times(m,m+1]\\
&\qquad \text{in } (\mathbb{T}_k-\{o\})\times\mathbb{R})\\
&\quad= P_{\delta,\lambda_c}(B)\prod_{\ell\leqslant 0,\, m\geqslant 1} P_{1,\lambda_c}(\{o\}\times(\ell-1,\ell] \nrightarrow \{o\}\times(m,m+1]\\
&\qquad \text{in } (\mathbb{T}_k-\{o\})\times\mathbb{R})\\
&\quad\geqslant P_{\delta,\lambda_c}(B)\prod_{\ell\leqslant 0,\, m\geqslant 1} (1-P_{1,\lambda_c}(\{o\}\times(\ell-1,l] \to \{o\}\times(m,m+1]))
\end{aligned}\tag{2.21}$$

Similar to (2.8), one can show by the FKG inequalities that for any $\{o\}\times t\in\{o\}\times(l-1,l]$ and $\{o\}\times u\in\{o\}\times(m,m+1]$

$$\begin{aligned}
&P_{1,\lambda}(\{o\}\times(\ell-1,\ell] \to \{o\}\times(m,m+1])\\
&\qquad\leqslant cP_{1,\lambda}(\{o\}\times(t-1) \to \{o\}\times(u+1))
\end{aligned}\tag{2.22}$$

with $c=1/P_{1,\lambda}$ (no death in $(\{o\}\times(\ell-2,\ell])\cup(\{o\}\times(m,m+2])$), which implies that

$$\begin{aligned}
&\sum_{\substack{\ell\leqslant 0\\ m\geqslant 1}} P_{1,\lambda}(\{o\}\times(\ell-1,\ell] \to \{o\}\times(m,m+1])\\
&\qquad\leqslant c\int_{-\infty}^{0}\int_{1}^{\infty} P_{1,\lambda}(\{o\}\times t \to \{o\}\times u)\,du\,dt
\end{aligned}\tag{2.23}$$

Using an argument similar to the proof of Lemma 2 in ref. 13 (which is built on an argument of ref. 6), it can be shown that when $\lambda_c < 1/(2\sqrt{k})$

$$\int_{-\infty}^{0} \int_{1}^{\infty} P_{1,\lambda_c}((o,t) \to (o,u))\, du\, dt < \infty \tag{2.24}$$

But $\lambda_c < 1/(2\sqrt{k})$ when $k > 4$ (see again, Theorem 2.2 of ref. 11). Combining (2.21), (2.23) and (2.24), we have that when $k > 4$

$$P_{\delta,\lambda_c}(\{o\} \times 1 \text{ is a breakpoint}) > 0$$

Therefore from the ergodicity and translation invariance along the time coordinate, when $k > 4$

$$P_{\delta,\lambda_c}(\{o\} \times n \text{ is a breakpoint for infinitely many (both positive and negative) } n) = 1 \tag{2.25}$$

Now suppose $\theta(\delta, \lambda_c) > 0$, then by (2.25) and arguing similar to (2.13), there exist an integer n and $x \in \mathbb{T}_k - \{o\}$ such that

$$P_{\delta,\lambda_c}(\text{infection initiated at } \{x\} \times n \text{ survives forever within } \mathbb{T}_k - \{o\}) > 0 \tag{2.26}$$

(2.26) also holds when δ is replaced by 1 since the event on its LHS does not depend on the death rate at $\{o\}$. A contradiction to $\theta(1, \lambda_c) = 0$. ∎

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